AN ANALOGUE OF A THEOREM DUE TO LEVIN AND VASCONCELOS

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ABSTRACT. Let $(R,\mathfrak{m})$ be a Noetherian local ring. Consider the notion of homological dimension of a module, denoted H-dim, for H= Reg, CI, CI_*, G, G^* or CM. We prove that, if for a finite R-module M of positive depth, $\text{H-dim}_R(\mathfrak{m}^i M)$ is finite for some $i \geq \operatorname{reg}(M)$, then the ring R has property H.

INTRODUCTION

One of the most influential results in commutative algebra is result of Auslander, Buchsbaum and Serre: A local ring $(R,\mathfrak{m})$ is regular if and only if $\operatorname{projdim}_R R/\mathfrak{m}$ is finite. This result is considerably strengthened by Levin and Vasconcelos. Their result can be read as follows,

Theorem (Levin-Vasconcelos [6]). *Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring. If there exists a finite module M such that $\operatorname{projdim}_R \mathfrak{m}^i M$ is finite, for some $i \geq 1$, then R is regular.*

Note that $\operatorname{projdim}_R 0$ is defined to be $-\infty$. This in particular implies that if $\dim R > 0$ then R is regular if and only if $\operatorname{projdim}_R(R/\mathfrak{m}^i)$ is finite for some $i \geq 1$.

On the other hand a variety of refinements of projective dimension are defined, namely G-dimension [1], CI-dimension [3], CI_*-dimension [4], G^*-dimension [11] and CM-dimension [4]. For the sake of uniformity of notation sometimes we write Reg-dimension for projective dimension. We say that R has property H with H=Reg (respectively, CI, CI_*, G^*, G or CM) if it is regular (respectively, complete intersection, complete intersection, Gorenstein, Gorenstein or Cohen-Macaulay). All the above homological dimensions generalise Auslander, Buchsbaum and Serre's result namely, the ring R has property H if and only if $\text{H-dim}_R R/\mathfrak{m}$ is finite.

So a natural question arises, whether it is possible to generalize Levin and Vasconcelos result to the above mentioned generalizations of projective dimension. We prove the result when $\operatorname{depth} M > 0$ and $n \gg 0$. To state the result we need an invariant $\rho(M)$, see 1.6. In [7] it is proved that $\rho(M) \leq \operatorname{reg}(M)$. Our main Theorem can be stated as follows.

Theorem 1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with residue field k and let M be a finite R-module of positive depth. Let H equal to Reg (respectively, CI,CI_*, G^*,G, or CM). If $\text{H-dim}_R(\mathfrak{m}^n M) < \infty$ for some $n \geq \rho(M)$, then R has property H.*

Date: February 1, 2008.
1991 *Mathematics Subject Classification.* Primary 13H05, Secondary 13D10.
Key words and phrases. Homological dimensions, Gorenstein dimension, CI dimension, CM-dimension, $\mathfrak{m}$-full ideal, superficial elements, Ratliff-Rush filtration .

As a corollary we get that if $\operatorname{depth} R > 0$ and $\text{H-dim}_R R/\mathfrak{m}^n$ is finite for some $n \geq \operatorname{reg}(R)$, then R has property H (see Corollary 4.3).

Recently Goto and Hayasaka [5] have proved that if G-dimension of an integrally closed $\mathfrak{m}$-primary ideal I is finite then R is Gorenstein. In [8] Iyengar and the second author used techniques used in the study of Hilbert functions to study some homological properties of $M/\mathfrak{m}^n M$ for $n \gg 0$. In this paper we combine some of the techniques used in these papers.

Here is an overview of the contents of the paper. In section 1 we introduce notation and discuss a few preliminary facts that we need. In Section two we define the notion of $\mathfrak{m}$-full submodules, which is a natural generalization of $\mathfrak{m}$-full ideal, defined by D. Rees (unpublished). See J. Watanabe's papers [12] and [13] for some properties of $\mathfrak{m}$-full ideals. In Section 3 we state the properties of H-dim that we need and prove a few that has not been explicitly proved before. In Section 4 we prove the Theorem 1 and a few corollaries.

1. Preliminaries

All rings are commutative Noetherian and all modules are finite. In this section we will give some auxiliary results that we need for the proof of Theorem 1. Let $(R, \mathfrak{m})$ be a local ring with residue field k and let M be a finite R-module. Let $G(R) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of R with respect to $\mathfrak{m}$ and let $G(M) = \bigoplus_{n\geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ be the associated graded module of M with respect to $\mathfrak{m}$. Set $\mathfrak{M} = \bigoplus_{n\geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ the maximal irrelevant ideal of $G(R)$.

1.1. Superficial elements: An element $x \in \mathfrak{m}$ is said to be M-*superficial* if there exists an integer $c > 0$ such that

$$(\mathfrak{m}^n M \colon_M x) \cap \mathfrak{m}^c M = \mathfrak{m}^{n-1} M \quad \text{for all} \quad n > c$$

When $\dim M > 0$ it is easy to see that x is superficial if and only if $x \notin \mathfrak{m}^2$ and x^* does not belong to any relevant associated prime of $G(M)$. Therefore, superficial elements always exist if the residue field k is infinite. If $\operatorname{depth} M > 0$ and x is M-superficial then $(\mathfrak{m}^n M \colon_M x) = \mathfrak{m}^{n-1}M$ for all $n \gg 0$ (see [10, p. 7] for $M = R$, the general case is similar).

1.2. Regularity: For $i \geq 0$ set $H^i_{\mathfrak{M}}(G(M))$ to be the i'th local cohomology of $G(M)$ with respect to $\mathfrak{M}$. The modules $H^i_{\mathfrak{M}}(G(M))$ are graded and Artininan. Define

$$\operatorname{reg}(M) = \max\{i + j \mid H^i_{\mathfrak{M}}(G(M))_j \neq 0\}.$$

1.3. If the residue field of R is finite then we resort to the standard trick to replace R by $R' = R[X]_S$ where $S = R[X] \setminus \mathfrak{m}R[X]$. The maximal ideal of R' is $\mathfrak{n} = \mathfrak{m}R'$. The residue field of R' is $l = k(X)$, the field of rational functions over k. Set $M' = M \otimes_R R'$. One can easily show that $\mathfrak{n}^i = \mathfrak{m}^i R'$ for all $i \geq 1$, $\dim M = \dim M'$, $\operatorname{depth} M = \operatorname{depth} M'$ and $\operatorname{reg}(M) = \operatorname{reg}(M')$.

1.4. Ratliff-Rush Filtration For every n we consider the chain of submodules of M

$$\mathfrak{m}^n M \subseteq \mathfrak{m}^{n+1} M \colon_M \mathfrak{m} \subseteq \mathfrak{m}^{n+2} M \colon_M \mathfrak{m}^2 \subseteq \cdots \subseteq \mathfrak{m}^{n+k} M \colon_M \mathfrak{m}^k \subseteq \cdots$$

This chain stabilizes at a submodule which we denote by

$$\widetilde{\mathfrak{m}^n M} = \bigcup_{k\geq 1} \mathfrak{m}^{n+k} M \colon_M \mathfrak{m}^k.$$

Clearly $\mathfrak{m}\widetilde{\mathfrak{m}^n M} \subseteq \widetilde{\mathfrak{m}^{n+1} M}$ and $\widetilde{\mathfrak{m}^{n+1} M} \subseteq \widetilde{\mathfrak{m}^n M}$. So $\mathcal{F} = \{\widetilde{\mathfrak{m}^n M}\}$ is an $\mathfrak{m}$-filtration of M. It is called the Ratliff-Rush filtration of M with respect to $\mathfrak{m}$.

1.5. If $\operatorname{depth} M > 0$ then the following holds (see [9] for the case $M = R$, see [7] where it is proved in general)

1. $\widetilde{\mathfrak{m}^n M} = \mathfrak{m}^n M$ for all $n \gg 0$.
2. $\operatorname{depth} G(M) > 0$ if and only if $\widetilde{\mathfrak{m}^n M} = \mathfrak{m}^n M$ for all $n \geq 0$.
3. If x is M-superficial then
   $(\widetilde{\mathfrak{m}^{n+1} M} \colon {}_M x) = \widetilde{\mathfrak{m}^n M}$ for all $n \geq 1$.

1.6. In view of property 1. and 2. above it is convenient to define the following invariant of M

$$\rho(M) = \min\{i \mid \widetilde{\mathfrak{m}^n M} = \mathfrak{m}^n M \quad \text{for all } n \geq i\}.$$

In [7, Theorem 5] it is proved that $\rho(M) \leq \operatorname{reg}(M)$.

2. $\mathfrak{m}$-full submodules

In this section we generalize to modules the notion of $\mathfrak{m}$-full ideals. Throughout this section R is local and M is a finite R-module.

Definition 2.1. A submodule N of M is called $\mathfrak{m}$-full if there exists $x \in \mathfrak{m}$ such that $\mathfrak{m}N \colon {}_M x = N$.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be local with infinite residue field and let M be an R-module of positive depth. Then for all n the submodule $\widetilde{\mathfrak{m}^n M}$ is $\mathfrak{m}$-full.*

Proof. Since the residue field is infinite we can choose $x \in \mathfrak{m}$ which is M-superficial with respect to $\mathfrak{m}$. For all n we have $\left(\widetilde{\mathfrak{m}^{n+1} M} \colon {}_M x\right) = \widetilde{\mathfrak{m}^n M}$. So we have

$$\widetilde{\mathfrak{m}^n M} \subseteq \left(\mathfrak{m}\widetilde{\mathfrak{m}^n M} \colon {}_M x\right) \subseteq \left(\widetilde{\mathfrak{m}^{n+1} M} \colon {}_M x\right) = \widetilde{\mathfrak{m}^n M}$$

So $\widetilde{\mathfrak{m}^n M}$ is $\mathfrak{m}$-full. $\square$

In the next proposition we collect the basic properties of $\mathfrak{m}$-full submodules.

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be local , M an R-module and N an $\mathfrak{m}$-full submodule of M. Let $x \in \mathfrak{m}$ be such that $\mathfrak{m}N \colon {}_M x = N$. We have the following:*

(1) $N \colon {}_M x = N \colon {}_M \mathfrak{m}$.
(2) *If $p_1, \ldots, p_l \in M$ is such that $\{\overline{p_1}, \ldots, \overline{p_l}\}$ is a basis of the k-vector space $(N \colon {}_M \mathfrak{m})/N$ then $xp_1, \ldots, xp_l$ form part of a minimal basis of N.*
(3) *Let $\{xp_1, \ldots, xp_l\} \bigcup \{z_1, \ldots, z_m\}$ generate N* minimally.
Define $\phi \colon N \to (N \colon {}_M \mathfrak{m})/N$ as follows: For $t = \sum_{i=1}^{l} a_i x p_i + \sum_{j=1}^{m} b_j z_j$, set $\phi(t) = \sum_{i=1}^{l} \overline{a_i p_i}$. The map ϕ is well defined and R-linear.
(4) *The R-linear map $\psi \colon (N \colon {}_M \mathfrak{m})/N \to N/xN$ defined by $\psi(s + N) = xs + xN$ is a split injection.*
(5) *k is a direct summand of N/xN*

Proof. (1) Clearly $N \colon {}_M x \supseteq N \colon {}_M \mathfrak{m}$. Note that

$$N \colon {}_M \mathfrak{m} = (\mathfrak{m}N \colon {}_M x) \colon {}_M \mathfrak{m} = (\mathfrak{m}N \colon {}_M \mathfrak{m}) \colon {}_M x \supseteq N \colon {}_M x.$$

So we get that $N \colon {}_M x = N \colon {}_M \mathfrak{m}$.

(2) If $\sum_{i=1}^{l} a_i x p_i \in \mathfrak{m}N$ then $\sum_{i=1}^{l} a_i p_i \in (\mathfrak{m}N \colon {}_M x) = N$. Since $\{\overline{p_1}, \dots, \overline{p_l}\}$ is a k-basis of $(N \colon {}_M \mathfrak{m})/N$ we get that $a_i \in \mathfrak{m}$ for all i.

(3) First note that if $t = \sum_{i=1}^{l} a_i x p_i + \sum_{j=1}^{m} b_j z_j$, set $u = \sum_{i=1}^{l} a_i p_i$. Since $xu = \sum_{i=1}^{l} a_i(xp_i) \in N$ we have that $u \in N \colon {}_M x = N \colon {}_M \mathfrak{m}$. Also note that if $t = \sum_{i=1}^{l} a_i x p_i + \sum_{j=1}^{m} b_j z_j = \sum_{i=1}^{l} a_i' x p_i + \sum_{j=1}^{m} b_j' z_j$ then $a_i - a_i' \in \mathfrak{m}$ for all i and so $\sum_{i=1}^{l} \left(\overline{a_i} - \overline{a_i'}\right) \overline{p_i} \in N$, since $p_i \in (N \colon {}_M \mathfrak{m})$ for all $i = 1, \dots, l$. So it follows that ϕ is well defined. Clearly ϕ is R-linear.

(4) Set $W = (N \colon {}_M \mathfrak{m})/N$. Clearly ψ is R-linear. Furthermore by (2) we have that ψ is injective. Note that the map $\phi \colon N \to W$ defined in (3) maps xN to 0 and so defines a map $\overline{\phi} \colon N/xN \to W$. Furthermore $\overline{\phi}\psi = id_W$. So ψ is split.

(5) Since $(N \colon {}_M \mathfrak{m})/N$ is a k-vector space, the assertion follows from (4). $\square$

3. SOME PROPERTIES OF H-DIM

Let $(R, \mathfrak{m})$ be a local ring and let M be a R-module. For all unexpalined terminology see the survey paper [2].

Remark 3.1. It suffices to prove Theorem 1 when H is equal to Reg (respectively, CI$_*$, G, or CM) since homological dimensions satisfy the inequalities

$$\text{CM-dim}_R M \le \text{G-dim}_R M \le \text{CI}_*\text{-dim}_R M \le \text{CI-dim}_R M \le \text{projdim}_R M$$

$$\text{G-dim}_R M \le \text{G}^*\text{-dim}_R M \le \text{CI-dim}_R M$$

Note that if any one of these dimensions is finite, then it is equal to those on its left (cf. [2, Theorem 8.8]).

We collect in Theorem 3.2 the properties of H-dim$_R$ when H is equal to Reg (respectively, CI$_*$, G, or CM) that are used in our proof of Theorem 1.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. For H equal to Reg (respectively, CI$_*$, G, or CM) the homological dimension* H-dim$_R M$ *has the following properties:*

1. *If x is M-regular and if* H-dim$_R M < \infty$ *then* H-dim$_R M/xM < \infty$.
2. *Let $0 \to N \to F \to M \to 0$ is an exact sequence with F a free R-module. If* H-dim$_R M < \infty$ *then* H-dim$_R N < \infty$.
3. *If N is a direct summand of M and if* H-dim$_R M < \infty$ *then* H-dim$_R N < \infty$.
4. *Set $R' = R[X]_{\mathfrak{m}R[X]}$ and $M' = M \otimes R'$. If* H-dim$_R M < \infty$ *then* H-dim$_R M' < \infty$.

Proof. We first state the results which were known before (See [2, 8.7]). Properties 1 and 2 are known for each of the homological dimensions Reg, CI$_*$, G, and CM. Property 3 was known for H = Reg and H = G. Finally property 4 was known for H equal to Reg, G and CI$_*$. Proposition 3.3 proves property 3 in the case H = CI$_*$ and CM. Proposition 3.4 proves property 4 in the case H = CM. $\square$

Proposition 3.3. *Let M be an R-module and let N be a direct summand of M. Then for H equal to Reg (respectively, CI$_*$, G, or CM) we have* H-dim$_R N \le$ H-dim$_R M$.

Proof. This result is known for H= Reg, G. Consider the case H=CM. We may assume that CM-dim$_R M$ is finite for otherwise there is nothing to prove. Let $R \to R' \leftarrow Q$ be the corresponding G-deformation. So G-dim$_Q M \otimes_R R' < \infty$ and

$\operatorname{CM-dim}_R M = \operatorname{G-dim}_Q M \otimes_R R' - \operatorname{G-dim}_Q R'$. Since $N \otimes_R R'$ is a direct summand of $M \otimes_R R'$ we have that $\operatorname{G-dim}_Q N \otimes_R R' \leq \operatorname{G-dim}_Q M \otimes_R R'$. In particular $\operatorname{CM-dim}_R N$ is finite. Therefore $\operatorname{CM-dim}_R N = \operatorname{G-dim}_Q N \otimes_R R' - \operatorname{G-dim}_Q R'$, So we also get $\operatorname{CM-dim}_R N \leq \operatorname{CM-dim}_R M$.

Now suppose H=CI$_*$ and assume that CI$*$-dim M is finite. Note that for a finite R-module L we have

$$\mathrm{CI}{*}\text{-}\dim_R L < \infty \quad \text{if and only if} \quad \operatorname{G-dim}_R(L) < \infty \text{ and } \operatorname{cx}_R L < \infty.$$

So $\operatorname{G-dim}_R(M) < \infty$ and $\operatorname{cx}_R M < \infty$. Since N is a direct summand of M we get that $\operatorname{G-dim}_R(N) < \infty$ and $\operatorname{cx}_R N < \infty$. Therefore $\mathrm{CI}{*}\text{-}\dim N$ is finite and it is easy to see that $\mathrm{CI}{*}\text{-}\dim N \leq \mathrm{CI}{*}\text{-}\dim M$. □

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Set $R' = R[X]_{\mathfrak{m}R[X]}$ and $M' = M \otimes R'$. we have* $\operatorname{CM-dim}_R M = \operatorname{CM-dim}_{R'} M'$.

To prove this Proposition we need the following Lemma.

Lemma 3.5. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local rings and let $\phi\colon R \to S$ be a flat local homomorphism. Set $R' = R[X]_{\mathfrak{m}R[X]}$ and $S' = S[X]_{\mathfrak{n}R[X]}$. Then the naturally induced $\phi' \colon R' \to S'$ is a flat local homomorphism.*

Proof. Clearly ϕ' is a local homomorphism. So it suffices to show ϕ' is flat. Note that ϕ induces a ring homomorphism $f\colon R[X] \to S[X]$. Since S is a flat R-module we have that $S[X] = S \otimes_R R[X]$ is a flat $R[X]$-module. So $S[X]_{\mathfrak{m}R[X]}$ is a flat $R' = R[X]_{\mathfrak{m}R[X]}$ module. Since S' is a further localisation of $S[X]_{\mathfrak{m}R[X]}$ we get that S' is a flat R'-algebra. □

Proof of Lemma 3.4. Note that R' is a faithfully flat extension of R. By [2, 8.7(6)] we have that $\operatorname{CM-dim}_R M \leq \operatorname{CM-dim}_{R'} M'$ with equality if $\operatorname{CM-dim}_{R'} M'$ is finite. If $\operatorname{CM-dim}_R M$ is infinite, then clearly $\operatorname{CM-dim}_{R'} M' = \infty$ and the result follows. So suppose $\operatorname{CM-dim}_R M$ is finite. To show the equality in this case it is enough to show that $\operatorname{CM-dim}_{R'} M'$ is finite. To this end we use definition 3.2 of [4], for CM-dimension. By this definition, if $\operatorname{CM-dim}_R M$ is finite, there exists a local flat extension $R \to S$ and a suitable S-module K, such that $G_K\text{-}\dim_S(M \otimes_R S)$ is finite. Since K is suitable, we have $\operatorname{Hom}_S(K, K) \cong S$ and $\operatorname{Ext}^i_S(K, K) = 0$, for all $i > 0$.

Consider the faithfully flat extension $S \to S' = S[X]_T$ where $T = S[X] \setminus \mathfrak{n}S[X]$ and X is an indeterminate over S. Since S' is a flat S-module, we have, $\operatorname{Hom}_{S'}(K \otimes_S S', K \otimes_S S') \cong S'$ and $\operatorname{Ext}^i_{S'}(K \otimes_S S', K \otimes_S S') = 0$ for all $i > 0$. So $K' = K \otimes_S S'$ is a suitable S'-module. Moreover using similar isomorphism, we can deduce that if $\mathrm{G}_K\text{-}\dim_S P = 0$ then $\mathrm{G}_{K'}\text{-}\dim_{S'} P \otimes_S S' = 0$. So $\mathrm{G}_{K'}\text{-}\dim_{S'} M \otimes_R S' < \infty$. This in conjunction with the fact that S' is a local flat extension of R' (See Lemma 3.5) implies that $\operatorname{CM-dim}_{R'}(M \otimes_R R')$ is finite. □

4. Generalized Levin-Vasconcelos Theorem

We prove the following:

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let M be a finite R-module of positive depth. If $\operatorname{H-dim}_R(\widetilde{\mathfrak{m}^n M}) < \infty$ for some n, then R has property H.*

An easy corollary of this theorem is Theorem 1.

Proof of Theorem 1. The result follows since for $n \geq \rho(M)$ we have that $\widetilde{\mathfrak{m}^n M} = \mathfrak{m}^n M$. □

Theorem 4.1 also has the following corollary:

Corollary 4.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k and let M be a finite R-module such that* $\operatorname{depth} G(M) > 0$. *If* $\text{H-dim}(\mathfrak{m}^n M) < \infty$ *for some n, then R has property H.*

Proof. Note that $\operatorname{depth} G(M) > 0$ implies $\operatorname{depth} M > 0$. Furthermore in this case $\widetilde{\mathfrak{m}^n M} = \mathfrak{m}^n M$ for all n. Therefore the result follows from Theorem 4.1. □

Proof of Theorem 4.1. As stated in Remark 3.1 for the proof of theorem it is only enough to consider the cases H= Reg, CI_*, G, CM. This we do.

Using the standard trick, in view of Theorem 3.2.4 we may assume that the residue field $R/\mathfrak{m}$ is infinite. So there exist $x \in \mathfrak{m}$ which is a M-superficial element. Set $N = \widetilde{\mathfrak{m}^n M}$. As it is shown in Proposition 2.2 the submodule N is $\mathfrak{m}$-full and in particular $\mathfrak{m}N \colon_M x = N$. Note that x is also N-regular. By Theorem 3.2.1 we have that $\text{H-dim}_R(N/xN)$ is finite. By Proposition 2.3(5) we have that k is a direct summand of N/xN. So by Theorem 3.2.3 we get that $\text{H-dim}\, k$ is finite. So R has property H. □

Using Theorem 1 and Theorem 3.2.2 we have the following corollary.

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a local ring with* $\operatorname{depth} R > 0$. *Let H equal to Reg (respectively, CI, CI_*, G^*, G, or CM). If* $\text{H-dim}_R(R/\mathfrak{m}^n) < \infty$ *for some $n \geq \rho(R)$, then R has property H.*

Acknowledgement

We wish to thank L.Avramov and Oana Veliche for carefully reading an earlier draft of this paper and for comments which have improved our exposition. The first author would like to thank School of Mathematics , TIFR for its hospitality.

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